Response of non-equilibrium systems at criticality: Ferromagnetic models in dimension two and above

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Abstract

We study the dynamics of ferromagnetic spin systems quenched from infinite temperature to their critical point. We show that these systems are aging in the long-time regime, i.e., their two-time autocorrelation and response functions and associated fluctuation-dissipation ratio are non-trivial scaling functions of both time variables. This is exemplified by the exact analysis of the spherical model in any dimension $D > 2$, and by numerical simulations on the two-dimensional Ising model. We show in particular that, for $1 \ll s$ (waiting time) $\ll t$ (observation time), the fluctuation-dissipation ratio possesses a non-trivial limit value X_∞, which appears as a dimensionless amplitude ratio, and is therefore a novel universal characteristic of non-equilibrium critical dynamics. For the spherical model, we obtain $X_\infty = 1 - 2/D$ for $2 < D < 4$, and $X_\infty = 1/2$ for $D > 4$ (mean-field regime). For the two-dimensional Ising model we measure $X_\infty \approx 0.26 \pm 0.01$.

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1 Introduction

Consider a ferromagnetic model, without quenched randomness, evolving from a disordered initial state, according to some dynamics at fixed temperature T. In the high-temperature paramagnetic phase ($T > T_c$), the system relaxes exponentially to equilibrium. At equilibrium, two-time quantities such as the autocorrelation function $C(t, s)$ or the response function $R(t, s)$ only depend on the time difference $\tau = t - s$, where s (waiting time) is smaller than t (observation time), and both quantities are simply related to each other by the fluctuation-dissipation theorem

$$R_{\mathrm{eq}}(\tau) = -\frac{1}{T}\,\frac{\mathrm{d}C_{\mathrm{eq}}(\tau)}{\mathrm{d}\tau}. \tag{1.1}$$

In the low-temperature phase ($T < T_c$) the system undergoes phase ordering. In this non-equilibrium situation, $C(t, s)$ and $R(t, s)$ are non-trivial functions of both time variables, which only depend on their ratio at late times, i.e., in the self-similar domain growth (or coarsening) regime [1]. This behavior is usually referred to as aging [2]. Moreover, no such simple relation as eq. (1.1) holds between correlation and response, i.e., $R(t, s)$ and $\partial C(t, s)/\partial s$ are no longer proportional. It is then natural to characterize the distance to equilibrium of an aging system by the so-called fluctuation-dissipation ratio [2, 3, 4]

$$X(t, s) = \frac{T\,R(t, s)}{\dfrac{\partial C(t, s)}{\partial s}}. \tag{1.2}$$

In recent years, several works [2, 3, 4, 5, 6, 7, 8, 9, 10] have been devoted to the study of the fluctuation-dissipation ratio for systems exhibiting domain growth, or for aging systems such as glasses and spin glasses, showing that in the low-temperature phase $X(t, s)$ turns out to be a non-trivial function of its two arguments. In particular, for domain-growth systems, analytical and numerical studies indicate that the limit fluctuation-dissipation ratio,

$$X_\infty = \lim_{s\to\infty} \lim_{t\to\infty} X(t, s), \tag{1.3}$$

vanishes throughout the low-temperature phase [6, 7, 8].

However, to date, only very little attention has been devoted to the response function $R(t, s)$, and fluctuation-dissipation ratio $X(t, s)$, for non-equilibrium systems *at criticality*. From now on, we will only have in mind ferromagnetic systems without quenched randomness. For instance one may wonder whether there exists, for a given model, a well-defined limit X_∞ at $T = T_c$, different from its trivial value $X_\infty = 0$ in the low-temperature phase, and to what extent X_∞ is universal. Indeed a priori, for a system such as a ferromagnet, quenched from infinitely high temperature to its critical point,

the limit fluctuation-dissipation ratio X_∞ at $T = T_c$ (if it exists) may take any value between $X_\infty = 1$ ($T > T_c$: equilibrium) and $X_\infty = 0$ ($T < T_c$: domain growth).

The only cases of critical systems for which the fluctuation-dissipation ratio has been considered are, to our knowledge, the models of ref. [3] (random walk, free Gaussian field, and two-dimensional X-Y model at zero temperature) which share the limit fluctuation-dissipation ratio $X_\infty = 1/2$, and the backgammon model, a mean-field model for which $T_c = 0$, where it has been shown that $X_\infty = 1$, up to a large logarithmic correction, for both energy fluctuations and density fluctuations [11, 12].

In a recent companion paper [13], we have determined the fluctuation-dissipation ratio $X(t, s)$ for the Glauber-Ising chain, another model for which $T_c = 0$. In particular, the limit fluctuation-dissipation ratio was found to be $X_\infty = 1/2$ (see also ref. [14]). In the present work we investigate the non-equilibrium correlation and response functions and the associated fluctuation-dissipation ratio in generic ferromagnetic models at their critical point. We first present (in section 2) an analytical study of the spherical model in arbitrary dimension. We then turn (in section 3) to a scaling analysis of the generic case, and to numerical simulations on the two-dimensional Ising model. One salient outcome of these joint works is the realization that the limit fluctuation-dissipation ratio X_∞ is a novel universal characteristic of critical dynamics, intrinsically related to non-equilibrium initial situations.

The present paper is written in a self-contained fashion. For the spherical model, though our main intention lies in the study of non-equilibrium dynamics at the critical point, we shall present the three situations $T > T_c$, $T = T_c$, and $T < T_c$ in parallel. The latter case has already been the subject of a number of investigations, for both correlation and response [5, 10, 1]. Results on the scaling behavior of the two-time auto-correlation function at T_c can be found in ref. [15]. For the two-dimensional Ising model, several numerical works have already been devoted to its non-equilibrium dynamics in the low-temperature phase, concerning both correlations [1, 16] and response [7]. We will therefore restrict our numerical study to the dynamics at the critical point.

2 The spherical model

2.1 Langevin dynamics

The ferromagnetic spherical model was introduced by Berlin and Kac [17], as an attempt to simplify the Ising model. It is solvable in any dimension, yet possesses non-trivial critical properties [17, 18]. Consider a lattice of points of arbitrary dimension D, chosen

to be hypercubic for simplicity, with unit lattice spacing. The spins $S_{\mathbf{x}}$, situated at the lattice vertices $\mathbf{x}$, are real variables subject to the constraint

$$\sum_{\mathbf{x}} S_{\mathbf{x}}^2 = N, \tag{2.1}$$

where N is the number of spins in the system. The Hamiltonian of the model reads

$$\mathcal{H} = -\sum_{(\mathbf{x},\mathbf{y})} S_{\mathbf{x}} S_{\mathbf{y}}, \tag{2.2}$$

where the sum runs over pairs of neighboring sites.

Throughout the following, we assume that the system is homogeneous, i.e., invariant under spatial translations. This holds for a finite sample with periodic boundary conditions, and (at least formally) for the infinite lattice. We also assume that the initial state of the system at $t = 0$ is the infinite-temperature equilibrium state. This state is fully disordered, in the sense that spins are uncorrelated. The dynamics of the system is given by the stochastic differential Langevin equation

$$\frac{\mathrm{d}S_{\mathbf{x}}}{\mathrm{d}t} = \sum_{\mathbf{y}(\mathbf{x})} S_{\mathbf{y}} - \lambda(t) S_{\mathbf{x}} + \eta_{\mathbf{x}}(t). \tag{2.3}$$

The first term, where $\mathbf{y}(\mathbf{x})$ denotes the $2D$ first neighbors of the site $\mathbf{x}$, is equal to the gradient $-\partial\mathcal{H}/\partial S_{\mathbf{x}}$, while $\lambda(t)$ is a Lagrange multiplier ensuring the constraint (2.1), which we choose to parameterize as

$$\lambda(t) = 2D + z(t), \tag{2.4}$$

and $\eta_{\mathbf{x}}(t)$ is a Gaussian white noise with correlation

$$\langle \eta_{\mathbf{x}}(t)\eta_{\mathbf{y}}(t')\rangle = 2T\,\delta_{\mathbf{x},\mathbf{y}}\,\delta(t-t'). \tag{2.5}$$

Equation (2.3) can be solved in Fourier space. Defining the spatial Fourier transform by the formulas

$$f^{\mathrm{F}}(\mathbf{q}) = \sum_{\mathbf{x}} f_{\mathbf{x}}\,\mathrm{e}^{-\mathrm{i}\mathbf{q}.\mathbf{x}}, \qquad f_{\mathbf{x}} = \int \frac{\mathrm{d}^D\mathbf{q}}{(2\pi)^D} f^{\mathrm{F}}(\mathbf{q})\,\mathrm{e}^{\mathrm{i}\mathbf{q}.\mathbf{x}}, \tag{2.6}$$

where

$$\int \frac{\mathrm{d}^D\mathbf{q}}{(2\pi)^D} = \int_{-\pi}^{\pi} \frac{\mathrm{d}q_1}{2\pi} \cdots \int_{-\pi}^{\pi} \frac{\mathrm{d}q_D}{2\pi} \tag{2.7}$$

is the normalized integral over the first Brillouin zone, we obtain

$$\frac{\partial S^{\mathrm{F}}(\mathbf{q},t)}{\partial t} = -[\omega(\mathbf{q}) + z(t)]S^{\mathrm{F}}(\mathbf{q},t) + \eta^{\mathrm{F}}(\mathbf{q},t), \tag{2.8}$$

where

$$\omega(\mathbf{q}) = 2\sum_{a=1}^{D}(1-\cos q_a) \underset{\mathbf{q}\to\mathbf{0}}{\approx} \mathbf{q}^2, \tag{2.9}$$

and

$$\langle\eta^{\mathrm{F}}(\mathbf{q},t)\eta^{\mathrm{F}}(\mathbf{q}',t')\rangle = 2T\,(2\pi)^D\,\delta^D(\mathbf{q}+\mathbf{q}')\,\delta(t-t'). \tag{2.10}$$

The solution to eq. (2.8) reads

$$S^{\mathrm{F}}(\mathbf{q},t) = \mathrm{e}^{-\omega(\mathbf{q})t-Z(t)}\left(S^{\mathrm{F}}(\mathbf{q},t=0) + \int_0^t \mathrm{e}^{\omega(\mathbf{q})t_1+Z(t_1)}\eta^{\mathrm{F}}(\mathbf{q},t_1)\,\mathrm{d}t_1\right), \tag{2.11}$$

with

$$Z(t) = \int_0^t z(t_1)\,\mathrm{d}t_1. \tag{2.12}$$

2.2 Equal-time correlation function

Our first goal is to compute the equal-time correlation function

$$C_{\mathbf{x}-\mathbf{y}}(t) = \langle S_{\mathbf{x}}(t)S_{\mathbf{y}}(t)\rangle, \tag{2.13}$$

which is a function of the separation $\mathbf{x}-\mathbf{y}$, by translational invariance. We have in particular

$$C_{\mathbf{0}}(t) = \langle S_{\mathbf{x}}(t)^2\rangle = 1, \tag{2.14}$$

because of the spherical constraint (2.1), and

$$C_{\mathbf{x}}(t=0) = \delta_{\mathbf{x},\mathbf{0}}, \tag{2.15}$$

reflecting the absence of correlations in the initial state. In eq. (2.13), the brackets denote the average over the ensemble of infinite-temperature initial configurations and over the thermal histories (realizations of the noise).

In Fourier space the equal-time correlation function is defined by

$$\langle S^{\mathrm{F}}(\mathbf{q},t)S^{\mathrm{F}}(\mathbf{q}',t)\rangle = (2\pi)^D\,\delta^D(\mathbf{q}+\mathbf{q}')\,C^{\mathrm{F}}(\mathbf{q},t). \tag{2.16}$$

Using the expression (2.11), averaging it over the white noise $\eta^{\mathrm{F}}(\mathbf{q},t)$ with variance given by eq. (2.10), and imposing the condition

$$C^{\mathrm{F}}(\mathbf{q},t=0) = 1 \tag{2.17}$$

implied by eq. (2.15), we obtain

$$C^{\mathrm{F}}(\mathbf{q},t) = \mathrm{e}^{-2\omega(\mathbf{q})t-2Z(t)}\left(1 + 2T\int_0^t \mathrm{e}^{2\omega(\mathbf{q})t_1+2Z(t_1)}\,\mathrm{d}t_1\right). \tag{2.18}$$

At this point, we are naturally led to introduce two functions, $f(t)$ and $g(T,t)$, which play a central role in the following developments.

The function $f(t)$ is explicitly given by

$$f(t) = \int \frac{\mathrm{d}^D\mathbf{q}}{(2\pi)^D}\,\mathrm{e}^{-2\omega(\mathbf{q})t} = \left(\mathrm{e}^{-4t} I_0(4t)\right)^D \underset{t\to\infty}{\approx} (8\pi t)^{-D/2}, \tag{2.19}$$

where

$$I_0(z) = \int \frac{\mathrm{d}q}{2\pi}\,\mathrm{e}^{z\cos q} \underset{z\to\infty}{\approx} (2\pi z)^{-1/2}\,\mathrm{e}^{z} \tag{2.20}$$

is the modified Bessel function.

The function

$$g(T,t) = \mathrm{e}^{2Z(t)} \tag{2.21}$$

is related to $f(t)$ by the constraint (2.14), namely

$$\int \frac{\mathrm{d}^D\mathbf{q}}{(2\pi)^D}\,C^{\mathrm{F}}(\mathbf{q},t) = \frac{1}{g(T,t)}\left(f(t) + 2T\int_0^t f(t-t_1)g(T,t_1)\,\mathrm{d}t_1\right) = 1, \tag{2.22}$$

which yields a linear Volterra integral equation for $g(T,t)$ [5], namely

$$g(T,t) = f(t) + 2T\int_0^t f(t-t_1)g(T,t_1)\,\mathrm{d}t_1. \tag{2.23}$$

This equation can be solved using temporal Laplace transforms, denoted by

$$f^{\mathrm{L}}(p) = \int_0^\infty f(t)\,\mathrm{e}^{-pt}\,\mathrm{d}t. \tag{2.24}$$

We obtain

$$g^{\mathrm{L}}(T,p) = \frac{f^{\mathrm{L}}(p)}{1-2Tf^{\mathrm{L}}(p)}, \tag{2.25}$$

with

$$f^{\mathrm{L}}(p) = \int \frac{\mathrm{d}^D\mathbf{q}}{(2\pi)^D}\,\frac{1}{p+2\omega(\mathbf{q})}. \tag{2.26}$$

The dependence of $g^{\mathrm{L}}(T,p)$ on temperature appears explicitly in eq. (2.25).

We now present an analysis of the long-time behavior of the function $g(T,t)$, considering successively the paramagnetic phase ($T > T_c$), the ferromagnetic phase ($T < T_c$), and the critical point ($T = T_c$). To do so, we shall extensively utilize eq. (2.25). We therefore investigate first the function $f^{\mathrm{L}}(p)$, as given in eq. (2.26). This function has no closed-form expression, except in one and two dimensions:

$$\begin{aligned} D=1: &\qquad f^{\mathrm{L}}(p) = \frac{1}{\sqrt{p(p+8)}}, \\ D=2: &\qquad f^{\mathrm{L}}(p) = \frac{2}{\pi|p+8|}\,\mathbf{K}\!\left(\frac{8}{|p+8|}\right), \end{aligned} \tag{2.27}$$

where $\mathbf{K}$ is the complete elliptic integral. Together with the definition (2.9), eq. (2.26) implies that $f^{\mathrm{L}}(p)$ is analytic in the complex p-plane cut along the real interval $[-8D, 0]$. The behavior of $f^{\mathrm{L}}(p)$ in the vicinity of the branch point at $p = 0$ can be analyzed heuristically as follows. The asymptotic behavior of $f(t)$ given in eq. (2.19) suggests that its Laplace transform has a universal singular part:

$$f^{\mathrm{L}}_{\mathrm{sg}}(p) \underset{p\to 0}{\approx} (8\pi)^{-D/2}\Gamma(1-D/2)p^{D/2-1}, \tag{2.28}$$

while there also exists a regular part of the form

$$f^{\mathrm{L}}_{\mathrm{reg}}(p) = A_1 - A_2 p + A_3 p^2 + \cdots, \tag{2.29}$$

where

$$A_k = \int \frac{\mathrm{d}^D\mathbf{q}}{(2\pi)^D}\,\frac{1}{(2\omega(\mathbf{q}))^k} \tag{2.30}$$

are non-universal (lattice-dependent) numbers, given in terms of integrals which are convergent for $D - 2k > 0$. For instance A_1 only exists for $D > 2$, and so on. Equations (2.28) and (2.29) jointly determine the small-p behavior of $f^{\mathrm{L}}(p)$, as a function of the dimensionality D:

$$\begin{aligned} D < 2: &\qquad f^{\mathrm{L}}(p) \approx (8\pi)^{-D/2}\Gamma(1-D/2)p^{-(1-D/2)}, \\ 2 < D < 4: &\qquad f^{\mathrm{L}}(p) \approx A_1 - (8\pi)^{-D/2}|\Gamma(1-D/2)|p^{D/2-1}, \\ D > 4: &\qquad f^{\mathrm{L}}(p) \approx A_1 - A_2 p. \end{aligned} \tag{2.31}$$

These expressions can be justified by more systematic studies (see e.g. ref. [19]): $f^{\mathrm{L}}(p)$ possesses an asymptotic expansion involving only powers of the form p^n and $p^{D/2-1+n}$, for $n = 0, 1, \ldots$ Whenever $D = 2, 4, \ldots$ is an even integer, the two sequences of exponents merge, giving rise to logarithmic corrections, which shall be discarded throughout the following.

In low enough dimension ($D < 2$), $f^{\mathrm{L}}(p)$ diverges as $p \to 0$. As a consequence, for any finite temperature, $g^{\mathrm{L}}(T, p)$ has a pole at some positive value of p, denoted by $1/\tau_{\mathrm{eq}}$, away from the cut of $f^{\mathrm{L}}(p)$. Hence

$$g(T, t) \underset{t\to\infty}{\sim} \mathrm{e}^{t/\tau_{\mathrm{eq}}}, \tag{2.32}$$

and therefore, as further analyzed below, the system relaxes exponentially fast to equilibrium, with a finite relaxation time τ_{eq}. The latter diverges as the zero-temperature phase transition is approached, as

$$\tau_{\mathrm{eq}} \underset{T\to 0}{\approx} \Big(2(8\pi)^{-D/2}\Gamma(1-D/2)T\Big)^{-2/(2-D)}. \tag{2.33}$$

In high enough dimension ($D > 2$), $f^{\mathrm{L}}(p = 0) = A_1$ is finite, so that the pole of $g^{\mathrm{L}}(T, p)$ hits the cut of $f^{\mathrm{L}}(p)$ at $p = 0$ at a finite critical temperature

$$T_c = \frac{1}{2A_1} = \left(\int \frac{\mathrm{d}^D \mathbf{q}}{(2\pi)^D} \frac{1}{\omega(\mathbf{q})}\right)^{-1}. \tag{2.34}$$

As $T \to T_c^+$, the relaxation time τ_{eq} diverges according to

$$\begin{aligned} 2 < D < 4: \qquad & \tau_{\mathrm{eq}} \underset{T \to T_c^+}{\approx} \left(\frac{2(8\pi)^{-D/2}|\Gamma(1 - D/2)|T_c^2}{T - T_c}\right)^{2/(D-2)}, \\ D > 4: \qquad & \tau_{\mathrm{eq}} \underset{T \to T_c^+}{\approx} \frac{2A_2 T_c^2}{T - T_c}. \end{aligned} \tag{2.35}$$

Note that these equations can be recast into the form $\tau_{\mathrm{eq}} \sim (T - T_c)^{-\nu z_c}$, where ν is the critical exponent of the correlation length, equal to $1/(D - 2)$ for $2 < D < 4$ and to $1/2$ for $D > 4$ [18], while z_c is the dynamic critical exponent, equal to 2 in the present case.[1]

We now discuss the asymptotic behavior of the function $g(T, t)$ according to temperature. Throughout the following, we will assume that $D > 2$, so that the model has a ferromagnetic transition at a finite T_c, given by eq. (2.34).

- In the paramagnetic phase ($T > T_c$), $g(T, t)$ still grows exponentially, according to eq. (2.32).
- In the ferromagnetic phase ($T < T_c$), a careful analysis of eq. (2.25) yields

  $$g(T, t) \underset{t \to \infty}{\approx} \frac{f(t)}{M_{\mathrm{eq}}^4} \approx \frac{(8\pi t)^{-D/2}}{M_{\mathrm{eq}}^4}, \tag{2.36}$$

  where the spontaneous magnetization M_{eq} is given by [18]

  $$M_{\mathrm{eq}}^2 = 1 - \frac{T}{T_c}. \tag{2.37}$$

- At the critical point ($T = T_c$), we obtain

$$\begin{aligned} 2 < D < 4: \qquad & g(T_c, t) \underset{t \to \infty}{\approx} (D - 2)(8\pi)^{D/2-1} \sin[(D - 2)\pi/2] \frac{t^{-(2-D/2)}}{T_c^2}, \\ D > 4: \qquad & g(T_c, t) \underset{t \to \infty}{\to} \frac{1}{4A_2 T_c^2}. \end{aligned} \tag{2.38}$$

[1] A summary of the values of static and dynamical exponents appearing in this work is given in Table 1.

Finally, eqs. (2.25), (2.26), (2.31), and (2.34) yield the following identities:

$$\begin{aligned}
\int_0^\infty f(t)\,\mathrm{d}t &= \frac{1}{2T_c}, \\
\int_0^\infty f(t)\,\mathrm{e}^{-t/\tau_{\mathrm{eq}}}\,\mathrm{d}t &= \frac{1}{2T} \qquad\qquad (T > T_c), \\
\int_0^\infty g(T,t)\,\mathrm{d}t &= \frac{1}{2T_c M_{\mathrm{eq}}^2} \qquad (T < T_c).
\end{aligned} \tag{2.39}$$

We are now in a position to discuss the temporal behavior of the equal-time correlation function in the different phases. Its expression (2.18) in Fourier space reads

$$C^{\mathrm{F}}(\mathbf{q},t) = \frac{\mathrm{e}^{-2\omega(\mathbf{q})t}}{g(T,t)}\left(1 + 2T\int_0^t \mathrm{e}^{2\omega(\mathbf{q})t_1} g(T,t_1)\,\mathrm{d}t_1\right), \tag{2.40}$$

using the definition (2.21) of $g(T,t)$. We shall consider in particular the dynamical susceptibility

$$\chi(t) = \frac{1}{T}\sum_{\mathbf{x}}\langle S_{\mathbf{0}}(t)S_{\mathbf{x}}(t)\rangle = \frac{C^{\mathrm{F}}(\mathbf{q}=\mathbf{0},t)}{T}, \tag{2.41}$$

for which eq. (2.40) yields

$$\chi(t) = \frac{1}{g(T,t)}\left(\frac{1}{T} + 2\int_0^t g(T,t_1)\,\mathrm{d}t_1\right). \tag{2.42}$$

The asymptotic expressions (2.32), (2.36), and (2.38) of $g(T,t)$ lead to the following predictions.

- In the paramagnetic phase ($T > T_c$), the correlation function converges exponentially fast to its equilibrium value, which has the Ornstein-Zernike form

$$C^{\mathrm{F}}_{\mathrm{eq}}(\mathbf{q}) = \frac{T}{\omega(\mathbf{q}) + \xi_{\mathrm{eq}}^{-2}}, \tag{2.43}$$

  where the equilibrium correlation length ξ_{eq} is given by

$$\xi_{\mathrm{eq}}^2 = 2\tau_{\mathrm{eq}}. \tag{2.44}$$

  The corresponding value of the equilibrium susceptibility is $\chi_{\mathrm{eq}} = \xi_{\mathrm{eq}}^2 = 2\tau_{\mathrm{eq}}$. Eq. (2.43) implies an exponential and isotropic fall-off of correlations, of the form $C_{\mathbf{x},\mathrm{eq}} \sim \mathrm{e}^{-|\mathbf{x}|/\xi_{\mathrm{eq}}}$, at large distances and for ξ_{eq} large, i.e., T close enough to T_c.

- In the ferromagnetic phase ($T < T_c$), using the third of the identities (2.39), we obtain a scaling form for the correlation function, namely

$$C^{\mathrm{F}}(\mathbf{q},t) \approx M_{\mathrm{eq}}^2\,(8\pi t)^{D/2}\,\mathrm{e}^{-2\mathbf{q}^2 t}, \tag{2.45}$$

or equivalently,

$$C_{\mathbf{x}}(t) \approx M_{\rm eq}^2\,{\rm e}^{-\mathbf{x}^2/(8t)}, \tag{2.46}$$

in the regime where $\mathbf{x}$ is large (i.e., $\mathbf{q}$ is small) and t is large. Both the Gaussian profile of the correlation function, and its scaling law involving one single diverging length scale

$$L(t) \sim t^{1/2}, \tag{2.47}$$

reflect the diffusive nature of the coarsening process. The growing length $L(t)$ can be interpreted as the characteristic size of an ordered domain. The dynamical susceptibility,

$$\chi(t) \approx \frac{M_{\rm eq}^2}{T}(8\pi t)^{D/2}, \tag{2.48}$$

grows as $\chi(t) \sim L(t)^D$, or else as the volume explored by a diffusive process.

- At the critical point ($T = T_c$), the equilibrium correlation function reads

$$C_{\rm eq}^{\rm F}(\mathbf{q}) \approx \frac{T_c}{\mathbf{q}^2}, \tag{2.49}$$

i.e.,

$$C_{\mathbf{x},{\rm eq}} \approx \frac{\Gamma(D/2-1)}{4\,\pi^{D/2}}\,\frac{T_c}{|\mathbf{x}|^{D-2}}. \tag{2.50}$$

These limiting expressions are reached according to scaling laws of the form

$$\begin{aligned} C^{\rm F}(\mathbf{q},t) &\approx C_{\rm eq}^{\rm F}(\mathbf{q})\,\Phi(\mathbf{q}^2 t), \\ C_{\mathbf{x}}(t) &\approx C_{\mathbf{x},{\rm eq}}\,\Psi(\mathbf{x}^2/t), \end{aligned} \tag{2.51}$$

with

$$\begin{aligned} 2<D<4: \qquad & \Phi(x) = 2x\int_0^1 {\rm e}^{-2x(1-z)}\,z^{D/2-2}\,{\rm d}z, \\ & \Psi(y) = {\rm e}^{-y/8}, \\ \\ D>4: \qquad & \Phi(x) = 1-{\rm e}^{-2x}, \\ & \Psi(y) = \frac{1}{\Gamma(D/2-1)}\int_{y/8}^{\infty} {\rm e}^{-z}\,z^{D/2-2}\,{\rm d}z. \end{aligned} \tag{2.52}$$

The second expression of eq. (2.51) has the general scaling form for the equal-time correlation function (see eq. (3.9)), with the known value of the static exponent of correlations $\eta = 0$ for the spherical model [18], and with $z_c = 2$, already found above.

The dynamical susceptibility grows linearly with time, as $\chi(t) \approx \Phi'(0)\,t$, i.e.,

$$\begin{aligned} 2<D<4: \qquad & \chi(t) \approx \frac{4}{D-2}\,t, \\ \\ D>4: \qquad & \chi(t) \approx 2t. \end{aligned} \tag{2.53}$$

2.3 Two-time correlation function

We now consider the two-time correlation function

$$C_{\mathbf{x}-\mathbf{y}}(t,s) = \langle S_{\mathbf{x}}(t) S_{\mathbf{y}}(s) \rangle, \tag{2.54}$$

with $0 \le s$ (waiting time) $\le t$ (observation time). Its Fourier transform $C^{\mathrm{F}}(\mathbf{q},t,s)$ is defined as in eq. (2.16). Using eq. (2.11), we obtain

$$C^{\mathrm{F}}(\mathbf{q},t,s) = \frac{\mathrm{e}^{-\omega(\mathbf{q})(t+s)}}{\sqrt{g(T,t)g(T,s)}} \left(1 + 2T \int_0^s \mathrm{e}^{2\omega(\mathbf{q})t_1} g(T,t_1)\,\mathrm{d}t_1 \right), \tag{2.55}$$

or else

$$C^{\mathrm{F}}(\mathbf{q},t,s) = C^{\mathrm{F}}(\mathbf{q},s)\,\mathrm{e}^{-\omega(\mathbf{q})(t-s)} \sqrt{\frac{g(T,s)}{g(T,t)}}, \tag{2.56}$$

using the expression (2.40) for $C^{\mathrm{F}}(\mathbf{q},s)$.

In the following, we shall be mostly interested in the two-time autocorrelation function

$$C(t,s) \equiv C_{\mathbf{0}}(t,s) = \langle S_{\mathbf{x}}(t) S_{\mathbf{x}}(s) \rangle = \int \frac{\mathrm{d}^D\mathbf{q}}{(2\pi)^D}\, C^{\mathrm{F}}(\mathbf{q},t,s), \tag{2.57}$$

for which eq. (2.55) yields

$$C(t,s) = \frac{1}{\sqrt{g(T,t)g(T,s)}} \left[f\left(\frac{t+s}{2}\right) + 2T \int_0^s f\left(\frac{t+s}{2} - t_1\right) g(T,t_1)\,\mathrm{d}t_1 \right]. \tag{2.58}$$

The autocorrelation with the initial state assumes the simpler form

$$C(t,s=0) = \frac{f(t/2)}{\sqrt{g(T,t)}}. \tag{2.59}$$

The asymptotic expressions (2.19), (2.32), (2.36), and (2.38) of the functions $f(t)$ and $g(T,t)$ lead to the following predictions.

- In the paramagnetic phase ($T > T_c$), as $s \to \infty$ with $\tau = t - s$ fixed, the system converges to its equilibrium state, where the correlation function only depends on τ:

  $$C(s+\tau,s) \underset{s\to\infty}{\longrightarrow} C_{\mathrm{eq}}(\tau) = T \int_\tau^\infty f(\tau_1/2)\,\mathrm{e}^{-\tau_1/(2\tau_{\mathrm{eq}})}\,\mathrm{d}\tau_1. \tag{2.60}$$

  This equilibrium correlation function decreases exponentially to zero as $\mathrm{e}^{-\tau/(2\tau_{\mathrm{eq}})}$ when $\tau \to \infty$. The initial value $C_{\mathrm{eq}}(\tau = 0) = 1$ is ensured by the second identity of eq. (2.39).

- In the ferromagnetic phase ($T < T_c$), two regimes need to be considered. In the first regime ($s \to \infty$ and τ fixed, i.e., $1 \sim \tau \ll s$), using again the identities (2.39), we obtain

$$C(s+\tau, s) \approx M_{\text{eq}}^2 + (1 - M_{\text{eq}}^2) C_{\text{eq,c}}(\tau), \tag{2.61}$$

where we have set

$$C_{\text{eq,c}}(\tau) = T_c \int_\tau^\infty f(\tau_1/2)\, \mathrm{d}\tau_1. \tag{2.62}$$

This function, which corresponds to the $T \to T_c$ limit of eq. (2.60), decreases only algebraically to zero when $\tau \to \infty$, as

$$C_{\text{eq,c}}(\tau) \underset{\tau\to\infty}{\approx} \frac{2(4\pi)^{-D/2}}{D-2}\, T_c\, \tau^{-(D/2-1)}, \tag{2.63}$$

as implied by eq. (2.19). The first identity of eq. (2.39) ensures that $C_{\text{eq,c}}(\tau = 0) = 1$.

In the second regime, where s and t are simultaneously large (i.e., $1 \ll s \sim \tau$), with arbitrary ratio

$$x = \frac{t}{s} = 1 + \frac{\tau}{s} \ge 1, \tag{2.64}$$

the correlation function obeys a scaling law of the form

$$C(t,s) \approx M_{\text{eq}}^2 \left(\frac{4ts}{(t+s)^2} \right)^{D/4} \approx M_{\text{eq}}^2 \left(\frac{4x}{(x+1)^2} \right)^{D/4}. \tag{2.65}$$

When $x \gg 1$, this expression behaves as

$$C(t,s) \approx A\, M_{\text{eq}}^2\, x^{-\lambda/2}, \tag{2.66}$$

which can be recast into

$$C(t,s) \sim M_{\text{eq}}^2 \left(\frac{L(t)}{L(s)} \right)^{-\lambda}, \tag{2.67}$$

where $L(t)$ is the length scale defined in eq. (2.47), and λ is the autocorrelation exponent (see section 3), which is equal to $D/2$ in the present case, in agreement with the result found in the $n \to \infty$ limit of the $O(n)$ model (see ref. [1], p. 386, and references therein).

Between these two regimes, the correlation function takes a plateau value

$$q_{\text{EA}} = \lim_{\tau\to\infty} \lim_{s\to\infty} C(s+\tau, s) = M_{\text{eq}}^2 = 1 - \frac{T}{T_c}, \tag{2.68}$$

known as the Edwards-Anderson order parameter (see e.g. ref. [20]).

Hereafter we shall refer to the first regime ($1 \sim \tau \ll s$) as the *stationary* regime, and to the second one ($1 \ll s \sim \tau$) as the *scaling* (or *aging*) regime. In the former, the system becomes stationary, though without reaching thermal equilibrium, because the system is coarsening. In the latter regime, as said above, the system is aging. It is possible to match these two kinds of behavior, corresponding respectively to eq. (2.61) and (2.65), into a single expression:

$$C(t = s + \tau, s) \approx (1 - M_{\text{eq}}^2) C_{\text{eq,c}}(\tau) + M_{\text{eq}}^2 \left(\frac{4ts}{(t+s)^2} \right)^{D/4}, \tag{2.69}$$

which is the sum of a term corresponding to the stationary contribution, and a term corresponding to the aging one. Let us finally recall that, in the context of glassy dynamics, in a low-temperature phase, the first regime, where $C(t, s) > q_{\text{EA}}$, is usually referred to as the β regime, while the second one, where $C(t, s) < q_{\text{EA}}$, is referred to as the α regime [2].

- At the critical point ($T = T_c$), the same two regimes are to be considered. However their physical interpretation is slightly different, since the order parameter M_{eq} vanishes, and symmetry between the phases is restored.

  In the first regime ($1 \sim \tau \ll s$), the system again becomes stationary, the autocorrelation function behaving as the $T \to T_c$ limit of eq. (2.60), that is

$$C(s + \tau, s) \underset{s\to\infty}{\longrightarrow} C_{\text{eq,c}}(\tau), \tag{2.70}$$

  which decreases algebraically to zero when $\tau \to \infty$ (cf. eq. (2.63)). In the second regime ($1 \ll s \sim \tau$), the correlation function obeys a scaling law of the form

$$C(t, s) \approx T_c \, s^{-(D/2-1)} \, F(x), \tag{2.71}$$

  where the scaling function $F(x)$ reads

$$\begin{aligned} 2 < D < 4: \qquad F(x) &= \frac{4(4\pi)^{-D/2}}{(D-2)(x+1)} \, x^{1-D/4} (x-1)^{1-D/2}, \\ D > 4: \qquad F(x) &= \frac{2(4\pi)^{-D/2}}{D-2} \left((x-1)^{1-D/2} - (x+1)^{1-D/2} \right). \end{aligned} \tag{2.72}$$

  In this regime the system is still aging, in the sense that $C(t, s)$ bears a dependence in both time variables. However, the scaling of expression (2.71) is different from that found in the low-temperature phase (see eq. (2.65)), which depends on the ratio $x = t/s$ only. The presence in eq. (2.71) of an additional s-dependence through the factor $s^{-(D/2-1)}$ can be interpreted as coming from the anomalous dimension of

the field $S_{\mathbf{x}}$ at T_c. In the critical region one has indeed $M_{\rm eq} \sim (T - T_c)^{\beta} \sim \xi_{\rm eq}^{-\beta/\nu}$. Replacing $\xi_{\rm eq}$ by s^{1/z_c} implies the replacement of $M_{\rm eq}^2$ by $s^{-2\beta/\nu z_c} \sim s^{-(D-2+\eta)/z_c}$. With $\eta = 0$ and $z_c = 2$, the factor $s^{-(D/2-1)}$ is thus recovered. Note that the static hyperscaling relation $2\beta/\nu = D - 2 + \eta$ holds for $D < 4$, while it is violated for $D > 4$ (see Table 1).

Two limiting regimes are of interest. First, for $x \to 1$, i.e., $1 \ll \tau \ll s$, eq. (2.71) matches eq. (2.63). Second, for $x \gg 1$, i.e., $1 \ll s \ll t$, one gets

$$F(x) \approx B\, x^{-\lambda_c/z_c}, \tag{2.73}$$

where the autocorrelation exponent λ_c (see section 3) is equal to $3D/2 - 2$ if $2 < D < 4$, and to D above four dimensions, in agreement with the result found in ref. [15].

We also quote for later reference the scaling law of the derivative

$$\frac{\partial C(t,s)}{\partial s} \approx T_c\, s^{-D/2}\, F_1(x), \tag{2.74}$$

with

$$F_1(x) = -\frac{D-2}{2}\, F(x) - x\, F'(x), \tag{2.75}$$

i.e.,

$$\begin{aligned}
2 < D < 4: \qquad & F_1(x) = (4\pi)^{-D/2} \frac{(D-2)(x+1)^2 + 2(x-1)^2}{(D-2)(x+1)^2}\, x^{1-D/4}(x-1)^{-D/2}, \\
D > 4: \qquad & F_1(x) = (4\pi)^{-D/2} \left((x-1)^{-D/2} + (x+1)^{-D/2} \right).
\end{aligned} \tag{2.76}$$

2.4 Two-time response function

Suppose now that the system is subjected to a small magnetic field $H_{\mathbf{x}}(t)$, depending on the site $\mathbf{x}$ and on time $t \geq 0$ in an arbitrary fashion. This amounts to adding to the ferromagnetic Hamiltonian (2.2) a time-dependent perturbation of the form

$$\delta\mathcal{H}(t) = -\sum_{\mathbf{x}} H_{\mathbf{x}}(t) S_{\mathbf{x}}(t). \tag{2.77}$$

The dynamics of the model is now given by the modified Langevin equation

$$\frac{\mathrm{d}S_{\mathbf{x}}}{\mathrm{d}t} = \sum_{\mathbf{y}(\mathbf{x})} S_{\mathbf{y}} - \lambda(t) S_{\mathbf{x}} + H_{\mathbf{x}}(t) + \eta_{\mathbf{x}}(t). \tag{2.78}$$

Causality and invariance under spatial translations imply that we have, to first order in the magnetic field $H_{\mathbf{x}}(t)$,

$$\langle S_{\mathbf{x}}(t)\rangle = \int_0^t \mathrm{d}s \sum_{\mathbf{y}} R_{\mathbf{x}-\mathbf{y}}(t,s) H_{\mathbf{y}}(s) + \cdots \tag{2.79}$$

This formula defines the two-time response function $R_{\mathbf{x}-\mathbf{y}}(t,s)$ of the model. A more formal definition reads

$$R_{\mathbf{x}-\mathbf{y}}(t,s) = \left.\frac{\delta\langle S_{\mathbf{x}}(t)\rangle}{\delta H_{\mathbf{y}}(s)}\right|_{\{H_{\mathbf{x}}(t)=0\}}. \tag{2.80}$$

The solution to eq. (2.78) reads, in Fourier space,

$$S^{\mathrm{F}}(\mathbf{q},t) = \mathrm{e}^{-\omega(\mathbf{q})t-Z(t)} \left(S^{\mathrm{F}}(\mathbf{q},t=0) + \int_0^t \mathrm{e}^{\omega(\mathbf{q})t_1+Z(t_1)} \left[H^{\mathrm{F}}(\mathbf{q},t_1) + \eta^{\mathrm{F}}(\mathbf{q},t_1)\right] \mathrm{d}t_1 \right). \tag{2.81}$$

It can be checked that the Lagrange function $\lambda(t)$, and hence $z(t)$ and $Z(t)$, remain unchanged, to first order in the magnetic field. As a consequence, the two-time response function reads, in Fourier transform,

$$R^{\mathrm{F}}(\mathbf{q},t,s) = \left.\frac{\delta\langle S^{\mathrm{F}}(\mathbf{q},t)\rangle}{\delta H^{\mathrm{F}}(\mathbf{q},s)}\right|_{\{H_{\mathbf{x}}(t)=0\}} = \mathrm{e}^{-\omega(\mathbf{q})(t-s)} \sqrt{\frac{g(T,s)}{g(T,t)}} \tag{2.82}$$

(cf. eq. (2.56)). In the following, we shall be mostly interested in the diagonal component of the response function, corresponding to coinciding points:

$$R(t,s) \equiv R_{\mathbf{0}}(t,s) = \left.\frac{\delta\langle S_{\mathbf{x}}(t)\rangle}{\delta H_{\mathbf{x}}(s)}\right|_{\{H_{\mathbf{x}}(t)=0\}} = \int \frac{\mathrm{d}^D\mathbf{q}}{(2\pi)^D} R^{\mathrm{F}}(\mathbf{q},t,s). \tag{2.83}$$

With the notations (2.19), (2.21), eq. (2.82) yields

$$R(t,s) = f\left(\frac{t-s}{2}\right) \sqrt{\frac{g(T,s)}{g(T,t)}}. \tag{2.84}$$

The response function at zero waiting time assumes the simpler form (cf. eq. (2.59))

$$R(t,s=0) = C(t,s=0) = \frac{f(t/2)}{\sqrt{g(T,t)}}. \tag{2.85}$$

The asymptotic expressions (2.19), (2.32), (2.36), and (2.38) of $F(t)$ and $g(T,t)$ lead to the following predictions.

- In the paramagnetic phase ($T > T_c$), at equilibrium, the response function only depends on τ, according to

  $$R_{\mathrm{eq}}(\tau) = f(\tau/2)\,\mathrm{e}^{-\tau/(2\tau_{\mathrm{eq}})}. \tag{2.86}$$

  Moreover, it is related to the equilibrium correlation function $C_{\mathrm{eq}}(\tau)$ of eq. (2.60) by the fluctuation-dissipation theorem (1.1), as it should.

- In the ferromagnetic phase ($T < T_c$), the two regimes defined in the previous section for the case of the autocorrelation function are still to be considered. In the stationary regime ($1 \sim \tau \ll s$), the response function behaves as the $T \to T_c$ limit of eq. (2.86), namely

$$R_{\mathrm{eq,c}}(\tau) = f(\tau/2) = -\frac{1}{T_c}\frac{\mathrm{d}C_{\mathrm{eq,c}}(\tau)}{\mathrm{d}\tau}, \tag{2.87}$$

  so that the fluctuation-dissipation theorem is valid.

  On the contrary, in the scaling regime ($1 \ll s \sim t$), the response function has the form

$$R(t,s) \approx (4\pi(t-s))^{-D/2}(t/s)^{D/4} = (4\pi s)^{-D/2}\,(x-1)^{-D/2}x^{D/4}, \tag{2.88}$$

  which, when compared to the corresponding expression (2.65) for the autocorrelation function, demonstrates the violation of the fluctuation-dissipation theorem (see section 2.5).

- At the critical point ($T = T_c$), in the stationary regime ($1 \sim \tau \ll s$), the response function still behaves as in eq. (2.87), so that the fluctuation-dissipation theorem still holds. In the scaling regime ($1 \ll s \sim t$), the response function obeys a scaling law of the form

$$R(t,s) \approx s^{-D/2}\,F_2(x), \tag{2.89}$$

  where the scaling function $F_2(x)$ reads

$$\begin{aligned} 2 < D < 4: &\quad F_2(x) = (4\pi)^{-D/2}x^{1-D/4}(x-1)^{-D/2},\\ D > 4: &\quad F_2(x) = (4\pi)^{-D/2}(x-1)^{-D/2}. \end{aligned} \tag{2.90}$$

  Again two limiting regimes are of interest. For $x \to 1$, the scaling result (2.90) matches eq. (2.87). For $x \gg 1$, one finds the same power-law fall-off for the functions $F(x)$, $F_1(x)$, and $F_2(x)$, that is

$$F(x) \sim F_1(x) \sim F_2(x) \sim x^{-\lambda_c/z_c} \tag{2.91}$$

  (see sections 2.5 and 3).

2.5 Fluctuation-dissipation ratio

As already mentioned in the introduction, the violation of the fluctuation-dissipation theorem (1.1) out of thermal equilibrium can be characterized by the fluctuation-dissipation ratio $X(t,s)$, defined in eq. (1.2). In the case of the spherical model, the results derived so far yield at once the following predictions.

- In the paramagnetic phase ($T > T_c$), the system converges to an equilibrium state, where the fluctuation-dissipation theorem holds. In other words, the fluctuation-dissipation ratio converges toward its equilibrium value

$$X_{\mathrm{eq}} = 1. \tag{2.92}$$

- In the ferromagnetic phase ($T < T_c$), the fluctuation-dissipation theorem (1.1) is only valid in the stationary regime ($1 \sim \tau \ll s$). On the contrary, in the scaling regime ($1 \ll s \sim \tau$), the results (2.65) and (2.88) imply that the fluctuation-dissipation ratio falls off as

$$X(t, s) \approx \frac{(8\pi)^{-D/2}}{D} \frac{4T}{M_{\mathrm{eq}}^2} \left(\frac{x+1}{x-1}\right)^{D/2+1} s^{-(D/2-1)}. \tag{2.93}$$

In particular, the limit fluctuation-dissipation ratio introduced in eq. (1.3) reads

$$X_\infty = 0. \tag{2.94}$$

- At the critical point ($T = T_c$), the scaling laws (2.74) and (2.89) imply that the fluctuation-dissipation ratio $X(t, s)$ becomes asymptotically a smooth function of the time ratio $x = t/s$:

$$X(t, s) \underset{t,s\to\infty}{\approx} \mathcal{X}(x) = \frac{F_2(x)}{F_1(x)}, \tag{2.95}$$

i.e., explicitly,

$$\begin{aligned} 2 < D < 4: \qquad & \mathcal{X}(x) = \frac{1}{1 + \dfrac{2}{D-2}\left(\dfrac{x-1}{x+1}\right)^2}, \\ D > 4: \qquad & \mathcal{X}(x) = \frac{1}{1 + \left(\dfrac{x-1}{x+1}\right)^{D/2}}. \end{aligned} \tag{2.96}$$

The scaling law (2.95) interpolates between the equilibrium behavior

$$\mathcal{X}(x) \underset{x\to 1}{\to} X_{\mathrm{eq}} = 1 \tag{2.97}$$

in the stationary regime of relatively short time differences, and a non-trivial limit value

$$\mathcal{X}(x) \underset{x\to\infty}{\to} X_\infty \tag{2.98}$$

at large time differences, given by

$$\begin{aligned} 2 < D < 4: \qquad & X_\infty = \frac{D-2}{D}, \\ D > 4: \qquad & X_\infty = \frac{1}{2}. \end{aligned} \tag{2.99}$$

Further comments on the scaling behavior of the fluctuation-dissipation ratio will be made in section 3.2.

3 The generic situation

3.1 Aging below T_c

Let us first briefly sketch the description of the dynamical behavior of a ferromagnetic system quenched from a disordered initial state to a temperature $T < T_c$ [21, 1, 5, 7, 8].

In the scaling regime ($1 \ll s \sim t$), the autocorrelation $C(t,s)$ is expected to be a function of the ratio $L(t)/L(s)$ only, where the length scale $L(t) \sim t^{1/z}$ is the characteristic size of an ordered domain, and z is the growth exponent, equal to 2 for non-conserved dynamics. More precisely,

$$C(t,s) = M_{\rm eq}^2\, f(t/s), \tag{3.1}$$

where the scaling function f is temperature independent. Furthermore we have, for $x = t/s \gg 1$, i.e., $1 \ll s \ll t$,

$$f(x) \approx A\, x^{-\lambda/z}, \tag{3.2}$$

where λ is the autocorrelation exponent [16]. For the spherical model, eqs. (2.65) and (2.67) match eqs. (3.1) and (3.2), with $\lambda = D/2$ and $z = 2$.

As a consequence, we have

$$\frac{\partial C(t,s)}{\partial s} \approx \frac{M_{\rm eq}^2}{s}\, f_1(x), \tag{3.3}$$

with $f_1(x) = -xf'(x)$, so that, when $x \gg 1$,

$$f_1(x) \approx A_1\, x^{-\lambda/z}, \tag{3.4}$$

with $A_1 = A\,\lambda/z$.

Although the situation of the response $R(t,s)$ is less clear-cut, it is however reasonable to make the scaling assumption

$$R(t,s) \approx s^{-1-a}\, f_2(x), \tag{3.5}$$

where $a > 0$ is an unknown exponent, and again with the behavior

$$f_2(x) \approx A_2\, x^{-\lambda/z} \tag{3.6}$$

when $x \gg 1$.

The scaling law (3.5) holds for the spherical model, with $a = D/2 - 1$, as can be seen from eq. (2.88). Furthermore, for non-conserved dynamics, at least in the case of a discrete broken symmetry, like e.g. in the Ising model, it has been argued [8, 22] that the integrated response $\rho(t,s)$ (to be defined in eq. (3.19)), scales as $\rho(t,s) \sim L(s)^{-1}\,\varphi(L(t)/L(s))$. This corresponds to eq. (3.5) with $a = 1/z = 1/2$.

The scaling laws (3.3), (3.5) imply

$$X(t,s) \approx s^{-a}\, g(x), \tag{3.7}$$

with $g(x) = (T/M_{\rm eq}^2)\, f_2(x)/f_1(x)$, in agreement with eq. (2.93) for the spherical model, and especially

$$X_\infty = 0. \tag{3.8}$$

3.2 Aging at T_c

Let us now turn to the situation where a ferromagnetic system is quenched from a disordered initial state to its critical point.

In such a circumstance, spatial correlations develop in the system, just as in the critical state, but only over a length scale which grows like t^{1/z_c}, where z_c is the dynamic critical exponent. For example the equal-time correlation function has the scaling form

$$C_{\mathbf{x}}(t) = |\mathbf{x}|^{-2\beta/\nu}\, \phi\Big(|\mathbf{x}|/t^{1/z_c}\Big)\,, \tag{3.9}$$

where β and ν are the usual static critical exponents. The scaling function $\phi(x)$ goes to a constant for $x \to 0$, while it falls off exponentially to zero for $x \to \infty$, i.e., on scales smaller than t^{1/z_c} the system looks critical, while on larger scales it is disordered. This behavior is illustrated in the case of the spherical model by eq. (2.51), corresponding to $2\beta/\nu = D - 2$ and $z_c = 2$ in eq. (3.9).

In the scaling region of the two-time plane, where both times s and t are large and comparable ($1 \ll s \sim t$), with arbitrary ratio $x = t/s$, the two-time autocorrelation function $C(t,s)$ is expected to obey a scaling law of the form (see the discussion below eq. (2.72), and ref. [15])

$$C(t,s) \approx s^{-2\beta/\nu z_c}\, F(x). \tag{3.10}$$

When both time scales are well separated ($1 \ll s \ll t$, i.e., $x \gg 1$), the scaling function $F(x)$ falls off as

$$F(x) \approx B\, x^{-\lambda_c/z_c}, \tag{3.11}$$

where λ_c is the critical autocorrelation exponent [23], related to the (magnetization) initial-slip critical exponent Θ_c [15] by $\lambda_c = D - z_c\Theta_c$.

We thus have

$$\frac{\partial C(t,s)}{\partial s} \approx s^{-1-2\beta/\nu z_c} F_1(x), \tag{3.12}$$

with $F_1(x) = -(2\beta/\nu z_c)F(x) - xF'(x)$, so that, when $x \gg 1$,

$$F_1(x) \approx B_1 \, x^{-\lambda_c/z_c}, \tag{3.13}$$

with

$$B_1 = \frac{\nu\lambda_c - 2\beta}{\nu z_c} \, B. \tag{3.14}$$

For the spherical model, eqs. (2.71), (2.73), and (2.74) respectively match eqs. (3.10), (3.11), and (3.12), with $\lambda_c = 3D/2 - 2$ if $D < 4$, and $\lambda_c = D$ if $D > 4$ (see Table 1).

The similarity between the results (2.74) and (2.89), obtained in the case of the spherical model, strongly suggests that $\partial C(t,s)/\partial s$ and $R(t,s)$ behave similarly in the generic case, i.e., one is lead to hypothesize that a scaling law of the form (3.12), with the same power-law fall-off (3.13), holds for the response, that is

$$R(t,s) \approx \frac{1}{T_c} \, s^{-1-2\beta/\nu z_c} F_2(x), \tag{3.15}$$

with, when $x \gg 1$,

$$F_2(x) \approx B_2 \, x^{-\lambda_c/z_c}. \tag{3.16}$$

The scaling laws (3.12) and (3.15) imply then that the fluctuation-dissipation ratio only depends on the time ratio x throughout the scaling region:

$$X(t,s) \approx \mathcal{X}(x) = \frac{F_2(x)}{F_1(x)}, \tag{3.17}$$

where the scaling function $\mathcal{X}(x)$ is universal. It appears indeed as a dimensionless combination of scaling functions. In turn, the hypothesis (3.16) implies that the limit fluctuation-dissipation ratio reads

$$X_\infty = \mathcal{X}(\infty) = \frac{B_2}{B_1}. \tag{3.18}$$

This number thus appears as a dimensionless amplitude ratio, in the usual sense of critical phenomena. It is therefore a novel universal quantity of non-equilibrium critical dynamics, as already claimed in ref. [13].

In the case of the spherical model, the analytical treatment of section 2 corroborates the above analysis, and yields the quantitative predictions (2.96) and (2.99).

In order to perform a numerical evaluation of X_∞, one needs to measure the response. A convenient way to do so is to measure instead the dimensionless integrated response function

$$\rho(t,s) = T \int_0^s R(t,u) \, \mathrm{d}u. \tag{3.19}$$

By eq. (2.79), this quantity is proportional to the thermoremanent magnetization $M_{\rm TRM}$, i.e., the magnetization of the system at time t obtained after applying a small magnetic field h, uniform and constant, between $t = 0$ and $t = s$:

$$M_{\rm TRM}(t, s) \approx \frac{h}{T}\,\rho(t, s). \tag{3.20}$$

The thermoremanent magnetization is a natural quantity to measure experimentally in spin glasses [2], and it is also accessible to numerical simulations, for systems with and without quenched randomness (see section 3.3).

The scaling law (3.15) for the response function implies

$$\rho(t, s) \approx s^{-2\beta/\nu z_c}\,F_3(x), \tag{3.21}$$

with $F_2(x) = -(2\beta/\nu z_c)F_3(x) - xF_3'(x)$, so that, when $x \gg 1$,

$$F_3(x) \approx B_3\,x^{-\lambda_c/z_c}, \tag{3.22}$$

with

$$B_3 = \frac{\nu z_c}{\nu\lambda_c - 2\beta}\,B_2. \tag{3.23}$$

A clear representation of the evolution of $X(t, s)$ in time is provided by the parametric plot of $\rho(t, s)$ against $C(t, s)$, obtained by varying t at fixed s [6, 7, 8]. For well-separated times in the scaling regime (i.e., $1 \ll s \ll t$), the common power-law behavior (3.11), (3.13), (3.16), and (3.22) implies that the limit fluctuation-dissipation ratio has the alternative expression

$$X_\infty = \frac{B_3}{B}, \tag{3.24}$$

which is equivalent to eq. (3.18), due to eqs. (3.14) and (3.23). In other words, the relationship (1.2) also holds in integral form, that is

$$\rho(t, s) \approx X_\infty\,C(t, s), \tag{3.25}$$

in the regime $1 \ll s \ll t$. The limit fluctuation-dissipation ratio can thus be measured as the slope of the parametric plot in the scaling region, i.e., near the origin of the $C - \rho$ plane. Eq. (3.25) is expected to hold as long as C and ρ are much smaller than the crossover scale

$$C^*(s) = C(2s, s) \sim s^{-2\beta/\nu z_c}, \tag{3.26}$$

corresponding to $\tau = s$. This quantity provides a measure of the size of the critical region, giving thus a quantitative definition of the critical analogue of $M_{\rm eq}^2$, involved in the discussion below eq. (2.72).

3.3 The two-dimensional Ising model: numerical simulations

In order to check the validity of the scaling analysis made in the previous section, beyond the case of the spherical model, we have performed numerical simulations on the ferromagnetic Ising model on the square lattice, evolving under heat-bath (Glauber) dynamics at its critical temperature $T_c = 2/\ln(1+\sqrt{2}) \approx 2.2692$, starting from a disordered initial state. The rules of the dynamics are as follows. Consider a finite system, consisting of $N = L^2$ spins $\sigma_{\mathbf{x}} = \pm 1$ situated at the vertices $\mathbf{x}$ of a square lattice, with periodic boundary conditions. The Ising Hamiltonian reads

$$\mathcal{H} = -\sum_{(\mathbf{x},\mathbf{y})} \sigma_{\mathbf{x}}\sigma_{\mathbf{y}}, \tag{3.27}$$

where the sum runs over pairs of neighboring sites. Heat-bath dynamics consists in updating the spins $\sigma_{\mathbf{x}}(t)$ according to the stochastic rule

$$\sigma_{\mathbf{x}}(t) \to \begin{cases} +1 & \text{with prob. } \dfrac{1+\tanh(h_{\mathbf{x}}(t)/T_c)}{2}, \\ -1 & \text{with prob. } \dfrac{1-\tanh(h_{\mathbf{x}}(t)/T_c)}{2}, \end{cases} \tag{3.28}$$

where the local field $h_{\mathbf{x}}(t)$ acting on $\sigma_{\mathbf{x}}(t)$ reads

$$h_{\mathbf{x}}(t) = \sum_{\mathbf{y}(\mathbf{x})} \sigma_{\mathbf{y}}(t), \tag{3.29}$$

with $\mathbf{y}(\mathbf{x})$ denoting the four neighbors of site $\mathbf{x}$.

Let us give a brief summary of known facts on the dynamics of the Ising model. For $T < T_c$, numerical studies have shown that the scaling forms (3.1) and (3.2) hold, with $z = 2$ (non-conserved dynamics) and $\lambda \approx 1.25$ [16]. The integrated response function (in another form, known as the ZFC magnetization) has been measured in ref. [7]. At $T = T_c$, the dynamic critical exponent reads $z_c \approx 2.17$ [24], and the autocorrelation exponent $\lambda_c \approx 1.59$ [23, 25].

Our aim is now to verify the hypotheses made in section 3.2, especially the scaling laws (3.15) and (3.21) for the response function, and to demonstrate the existence of a non-trivial limit X_∞.

Computing $C(t,s)$ with good statistics is rather easy, while the computation of $\rho(t,s)$ requires more effort. We have followed the lines of the method introduced in ref. [7]. In order to isolate the diagonal component of the response function, a quenched, spatially random magnetic field, is applied to the system from $t = 0$ to $t = s$. This magnetic field is of the form $H_{\mathbf{x}} = h_0\varepsilon_{\mathbf{x}}$, with a constant small amplitude h_0, and a quenched random modulation, $\varepsilon_{\mathbf{x}} = \pm 1$ with equal probability, independently at each site $\mathbf{x}$. The heat-bath

dynamical rules are modified by adding up the magnetic field $H_{\mathbf{x}}$ to the local field $h_{\mathbf{x}}(t)$ of eq. (3.29). We have then

$$\overline{\langle \varepsilon_{\mathbf{x}} \sigma_{\mathbf{x}}(t) \rangle} = h_0 \int_0^s R(t,u)\,\mathrm{d}u = \frac{h_0}{T}\,\rho(t,s) = M_{\mathrm{TRM}}(t,s), \tag{3.30}$$

where the bar means an average with respect to the distribution of the modulation $\varepsilon_{\mathbf{x}}$ of the magnetic field.

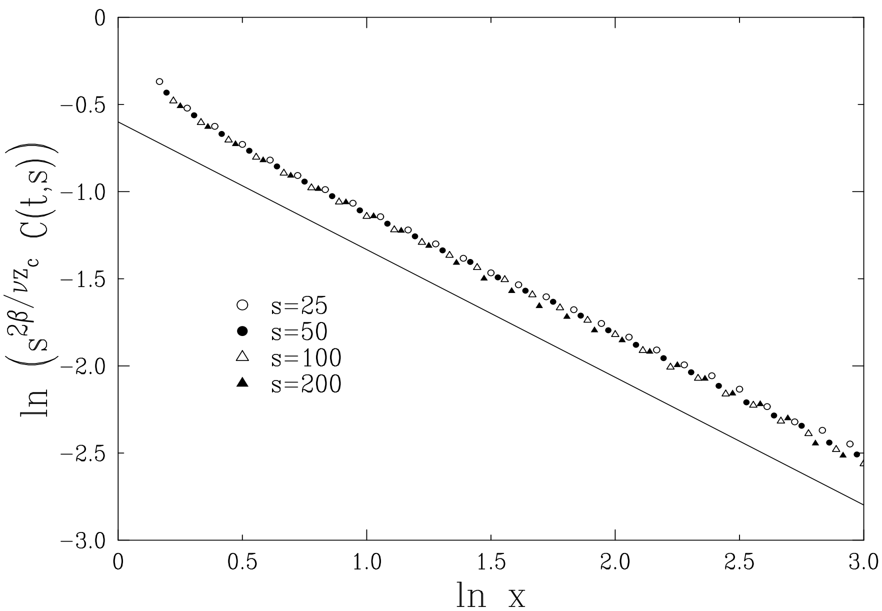

Figure 1: Log-log plot of the critical autocorrelation function $C(t,s)$ of the two-dimensional Ising model, against time ratio $x = t/s$, for several values of the waiting time s. Data are multiplied by $s^{2\beta/\nu z_c}$, in order to demonstrate collapse into the scaling function $F(x)$ of eq. (3.10). Straight line: exponent $-\lambda_c/z_c \approx -0.73$ of the fall-off at large x.

We have first checked the validity of the scaling laws (3.10), and especially (3.21). Figures 1 and 2 respectively show log-log plots of the autocorrelation function $C(t,s)$ and of the corresponding integrated response function $\rho(t,s)$, against the time ratio $x = t/s$, for several values of the waiting time s. For each value of s, the simulations are run up to $t/s = 10$, and data are averaged over at least 500 independent samples of size 300×300. For the response function, the amplitude of the quenched magnetic field reads $h_0 = 0.05$. Multiplying the data by $s^{2\beta/\nu z_c}$, with $2\beta/\nu z_c \approx 0.115$, gives good data collapse, thus producing a plot of the scaling functions $F(x)$ and $F_3(x)$. The data follow a power-law fall-off at large values of x, with a slope in good agreement with the value $-\lambda_c/z_c \approx -0.73$, shown on the plots as a straight line.

We then turned to an investigation of the parametric plot of these data in the $C-\rho$ plane. At the qualitative level, this plot, shown in Figure 3 for several values of the

waiting time s, confirms our expectations. The stationary regime ($1 \sim \tau \ll s$, i.e., roughly speaking, $C > C^*(s)$), corresponds to the right part of the plot. The symbols show the data for small integer values of the time difference, $\tau = t - s = 0, \ldots, 8$, illustrating the fast decay of correlation and integrated response in the stationary regime. The rightmost points, corresponding to $\tau = 0$, i.e., $C = C(s, s) = 1$, are compatible with the scaling law $1 - \rho(s, s) \sim C^*(s) \sim s^{-2\beta/\nu z_c}$. The validity of the fluctuation-dissipation theorem is testified by the unit slope of this part of the plot, shown as a full straight line. The aging regime ($1 \ll s \sim t$, i.e., roughly speaking, $C < C^*(s)$), corresponds to the left part of the plot. As expected, the data crossover toward a non-trivial slope, equal to the limit fluctuation-dissipation ratio X_∞. The dashed line shows the slope $X_\infty = 0.26$, obtained by the analysis described below.

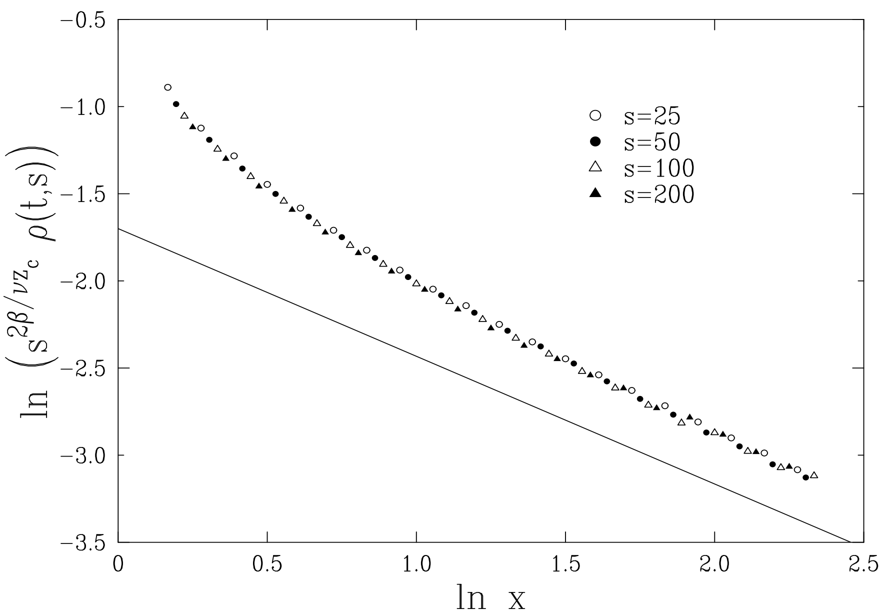

Figure 2: Log-log plot of the critical integrated response function $\rho(t, s)$ of the two-dimensional Ising model, against time ratio $x = t/s$, for several values of the waiting time s. Data are multiplied by $s^{2\beta/\nu z_c}$, in order to demonstrate collapse into the scaling function $F_3(x)$ of eq. (3.21). Straight line: exponent $-\lambda_c/z_c \approx -0.73$ of the fall-off at large x.

In order to obtain a quantitative prediction of the limit fluctuation-dissipation ratio X_∞, we have followed two approaches. Figure 4 depicts the local slope of the plot of Figure 3, i.e., the ratio ρ/C, against C, in the aging regime. The data for the largest available waiting time $s = 200$ have been discarded from the analysis because they appear as too noisy on that scale. The data look pretty linear all over the range presented in the plot. This precocious scaling is due to the fact that the exponent $2\beta/\nu z_c \approx 0.115$ is small.

Hence the size of the critical region, given by the estimate (3.26), is very large, at least for waiting times s accessible to computer simulations. We have indeed, for example, $C^*(100) = C(200, 100) \approx 0.24$. The straight lines show a constrained least-square fit of the three series of data, imposing a common intercept. The value of this intercept yields the prediction $X_\infty \approx 0.262$.

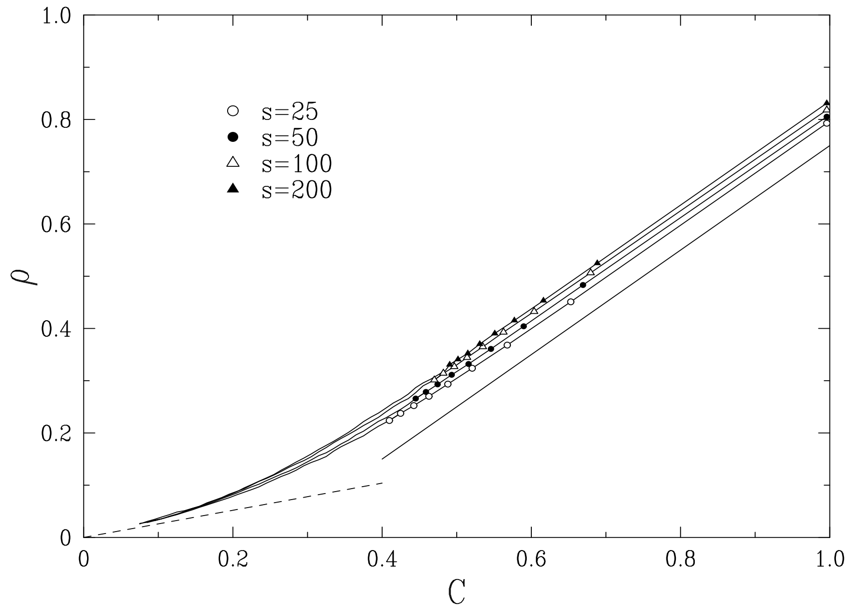

Figure 3: Parametric plot of the integrated response $\rho(t, s)$ against the autocorrelation $C(t, s)$, using the data of Figures 1 and 2. Symbols: data for integer time differences $\tau = t - s = 0, \ldots, 8$. Full line: unit slope corresponding to the fluctuation-dissipation theorem in the stationary regime. Dashed line: limit slope $X_\infty = 0.26$ (see text and Figures 4 and 5).

We have also followed an alternative approach, aiming at subtracting most of the deviations of the ratio ρ/C with respect to its limit X_∞ at $C \to 0$. This can be done by incorporating the known limit of the stationary regime, i.e., $\rho \approx 1$ as $C \to 1$, into a quadratic phenomenological formula: $\rho \approx X_\infty C + (1 - X_\infty)C^2$. This formula can be rewritten as $X_\infty \approx (\rho - C^2)/(C(1-C))$, suggesting to plot $(\rho - C^2)/(C(1-C))$ against C, instead of the mere ratio ρ/C. This has been done in Figure 5. As expected, the vertical scale has been considerably enlarged. In return this procedure increases the statistical noise on the data points. The straight lines again show a constrained least-square fit, yielding $X_\infty \approx 0.260$.

We can conclude from this numerical analysis that we have

$$X_\infty = 0.26 \pm 0.01 \qquad (3.31)$$

for the ferromagnetic Ising model in two dimensions.

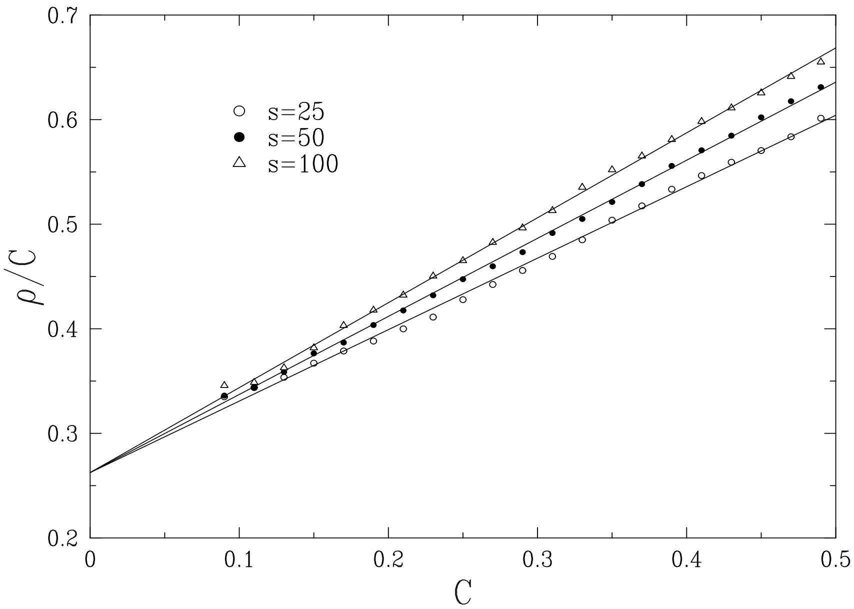

Figure 4: Parametric plot of the ratio ρ/C against C. Straight lines: constrained least-square fit with common intercept, yielding $X_\infty \approx 0.262$.

4 Discussion

In the present work we dealt with the dynamics of ferromagnetic spin systems quenched from infinite temperature to their critical state. This study, exemplified by the exact analysis of the spherical model in any dimension $D > 2$, and by numerical simulations on the two-dimensional Ising model, complements that of the Glauber-Ising chain, presented in a companion paper [13]. The main results obtained in this work can be summarized as follows.

In such a non-equilibrium situation, these systems are aging in the sense that their correlation and response functions depend non-trivially on the waiting time s as well as on the observation time t, whenever these two times are simultaneously large. The corresponding scaling laws (see eqs. (3.10), (3.12), (3.15), and (3.21)), involve powers of s, related to the static anomalous dimension of the magnetization, and universal scaling functions of the ratio $x = t/s$. In the regime of large time separations, i.e., $1 \ll s \ll t$ (or $x \gg 1$), these scaling functions fall off algebraically with the common exponent λ_c/z_c.

The fluctuation-dissipation ratio $X(t, s)$, characterizing the violation of the fluctuation-dissipation theorem, has a universal scaling form $\mathcal{X}(x)$, and, for well-separated times

in the aging regime, it assumes a limit value X_∞ equal to a dimensionless amplitude ratio (see eqs. (3.18) and (3.24)). Therefore, as announced in ref. [13], X_∞ is a novel universal characteristic of critical dynamics, which is intrinsically related to the non-equilibrium initial condition of a critical quench from a disordered state.

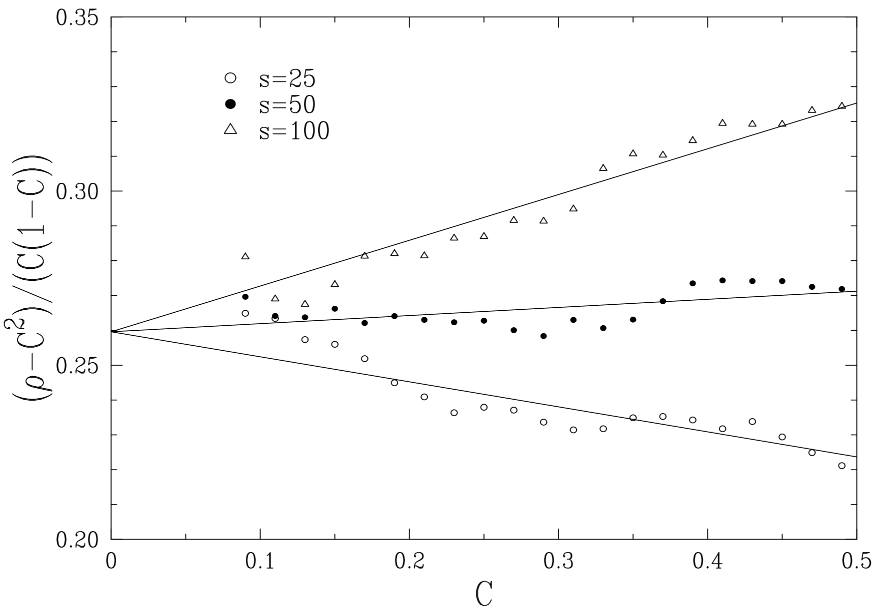

Figure 5: Parametric plot of the combination $(\rho - C^2)/(C(1-C))$ against C. Straight lines: constrained least-square fit yielding $X_\infty \approx 0.260$.

The ferromagnetic models studied in the present work turn out to have values of X_∞ in the range

$$0 \le X_\infty \le \frac{1}{2}. \tag{4.1}$$

We have indeed $X_\infty = 1 - 2/D$ if $2 < D < 4$, and $X_\infty = 1/2$ for $D > 4$, for the spherical model, and $X_\infty \approx 0.26 \pm 0.01$ for the two-dimensional Ising model. Let us mention that preliminary simulations on the three-dimensional Ising model yield $X_\infty \approx 0.40$. The backgammon, for which $X_\infty = 1$ [11, 12], thus belongs to another class of models.

The mean-field value

$$X_\infty^{\mathrm{MF}} = \frac{1}{2}, \tag{4.2}$$

obtained for the spherical model in dimension $D > 4$, also holds for a variety of models which are not mean-field-like, including the Glauber-Ising chain [13] and the two-dimensional X-Y model at zero temperature [3].

Let us finally discuss a few open questions. It would be interesting to know whether there is an analogue for the present case of the results found for models with discontinuous spin-glass transitions, where the violation of the fluctuation-dissipation theorem is

related to the configurational entropy [26]. One would also like to know the status of the quantity X_∞ for non-equilibrium systems with quenched disorder, or for systems defined by dynamical rules without detailed balance.

In principle the limit fluctuation-dissipation ratio X_∞ could be calculated by field-theoretical renormalization-group methods, generalizing the computations done for universal amplitude ratios in usual static critical phenomena [27], as series in either $\varepsilon = 4 - D$, or in $1/n$ for the n-component Heisenberg model, the spherical model corresponding to the $n \to \infty$ limit. The dimensionless time ratio $x = t/s$, appearing in the two-time autocorrelation and response functions and fluctuation-dissipation ratio, is a temporal analogue of aspect ratios, which play an important role in static critical phenomena and finite-size scaling theory [28]. One may therefore wonder whether the latter, and especially its latest developments involving conformal and modular invariance, could be used in order to put constraints on non-equilibrium critical dynamics. Generalized symmetry groups, such as those introduced in ref. [29], may also play a role in this issue.

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Table and caption

exponent	spherical ($2 < D < 4$)	spherical ($D > 4$)	Ising ($D = 2$)
η	0	0	1/4
β	1/2	1/2	1/8
ν	$1/(D-2)$	1/2	1
z	2	2	2
λ	$D/2$	$D/2$	≈ 1.25
z_c	2	2	≈ 2.17
λ_c	$3D/2 - 2$	D	≈ 1.59
Θ_c	$1 - D/4$	0	≈ 0.19

Table 1: Static and dynamical exponents of the ferromagnetic spherical model and of the two-dimensional Ising model. First group: usual static critical exponents η, β, and ν (equilibrium). Second group: zero-temperature dynamical exponents z and λ (coarsening below T_c). Third group: dynamic critical exponents z_c, λ_c, and Θ_c (non-equilibrium critical dynamics).

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